Vanda - Ligia DĂNĂILĂ, Irina IGNĂTESCU - MANEA

CONSIDERATIONS ON PLOTTING OF SOME PLANE CURVES

Abstract: *Plane curves can be analytically conveyed by means of a Cartesian equation, an implicit Cartesian equation, a parametric equation or a polar equation. The paper proposes an analysis regarding the way some plane curves can be plotted according to their analytic convey. Curves' plotting enables acknowledging the evolution of technological phenomena and processes, as well as it enables estimating the intensity, the minimum and maximum values capable of achieving.*

Key words: *plane curves, equations, mathematical functions, Cartesian co-ordinates, polar co-ordinates, function's graph.*

1. INTRODUCTION

A plane curve can be conveyed as follows: by means of a Cartesian equation (1), by means of an implicit Cartesian equation (4), by means of parametric equations (27), or by means of a polar equation (19) [1], [2] All these convey means are encountered in practice in the most various human activities: technique, economics, astronomy, biology and medicine, linguistics, everywhere mathematics is useful [5].

The concept of mathematical function came out in ancient Greece and it represented the reliance of geometric nature quantities. Isaac Newton and Gottfried Wilhelm Leibniz also used the concept of function as a correlation between various geometrical quantities. Johann Bernoulli (1718) defines the function without using graphic representation. For Leonhard Euler the concept of function was connected to the possibility of expressing quantities by means of formulas. The term of function was coined by G. W. Leibniz (1692) and the notation $f(x)$ was proposed by L. Euler (1734) [1].

If the cause a certain phenomenon depends on is denoted x and the law the studied phenomenon occurs is denoted $f(x)$, it may be written as (1). In this mathematical relationship f represents an analytical convey that can be a formula, a symbolic notation of elementary mathematical operations performed in a specific order with variable and constant quantities. The correlation of the two real variables x and y can be plotted on plane. A function's graph (1) is the set of all the points belonging to a plane co-ordinates x and y of which validates the equation (1). Therefore, in order to realize the manner the effect (y) evolves when the cause (x) takes various values, the graphic representation is often used [5].

The paper proposes an analysis over the manner some plane curves can be plotted according to their analytical convey.

2. PLOTTING OF CURVES CONVEYED BY MEANS OF A CARTESIAN EQUATION

Plane curves can be conveyed by means of a Cartesian equation as follows [4], [7]:

$$y = f(x) \tag{1}$$

In order to draw the curve adequate to such an equation few stages must be browsed:

- Settle the definition set of the function
  If the definition set of the function is not indicated it is implied as the set of all the points where the operations that define the function are logical.
  The function is analysed if it is even, odd or periodical. If the function is odd, it is enough to plot the graph for the positive set of values of x, because the plotting will be symmetrical in report to the origin of co-ordinates system. If the function in even, it is enough to plot the graph for the positive set of values of x, because the plotting will be symmetrical in report to OY axis. If the function is periodical, it is enough to plot the graph for values of x belonging to a range value, because function's graph will iterate.
  It shall be given to whether there are intersection points of the curve with the axes of co-ordinates system.
- Limits to endpoints, function's continuity, asymptotes
  If the definition set of the function is a reunion of increments, function's limits at the endpoints of each increment are calculated [2], [6].
  The horizontal, vertical and angled asymptotes are found, if there are.
  The set of points for which the function is continuous are found.
- The first-order derivative
  The set of points for which the function is differentiable are found and the derivative $f'(x)$ is calculated. For the positive values of the derivative the graph is strictly increasing, for the negative values of the derivative the graph is strictly decreasing, for zero value of the derivative the graph is steady.
- The second-order derivative (if there is)
  The set of points for which the function is twice differentiable are found and the derivative $f''(x)$ is calculated. For the positive values of the derivative f'' the graph is strictly convex, for the negative values of the derivative f'' the graph is strictly concave, for zero value of the derivative f'' the graph is linear.
- Function's variance table
  The results obtained after completing the mentioned stages are gathered in a table containing the notable

values of x, the intersections with the axes of co-ordinates system, the values adequate to derivative f', graph's minimum (m) and maximum (M) points, the values adequate to derivative f'' [8], [9].

- Drawing the graph
  The axes of the orthogonal system XOY are plotted and the co-ordinates of the notable points are measured.
  The asymptotes are drawn if there are.
  The notable points of the graph are connected by a continuous curve considering its characteristics (monotony, convexity, asymptotes).

To illustrate the method to draw a function's graph conveyed by means of a Cartesian equation two curves (Figure 1, Figure 2) are presented.

The curve plotted in Figure 1 is specified by equation:

$$y = a \sin x \qquad (2)$$

For the constant a the assigned values are: a = ¼; ½; 1; 2; 4. The x co-ordinate of the notable points has the values: x == 0; π/2; π; 3π/2; 2π. The maximum and minimum points are marked M, respectively m.

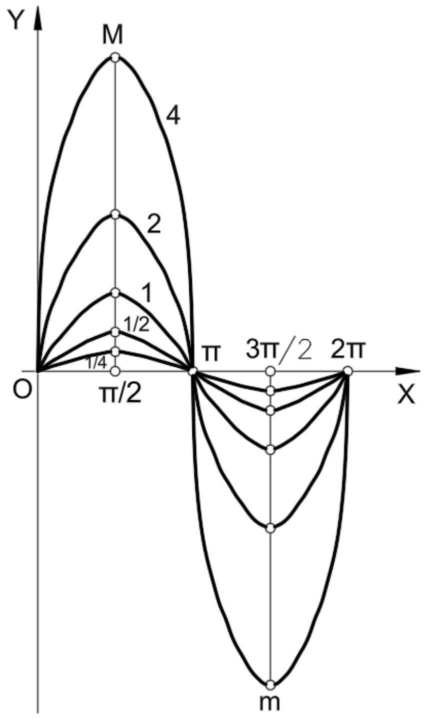

Figure 1 Graph 1 of a Cartesian equation

The curve plotted in Figure 2 is specified by equation:

$$y = a \cos x \qquad (3)$$

For the constant a the assigned values are: a = ½; 1; 2; 4. The x co-ordinate of the notable points has the values: x == - π; -π/2; 0; π/2; π; 3π/2; 2π. The maximum and minimum points are marked M, respectively m [8], [9].

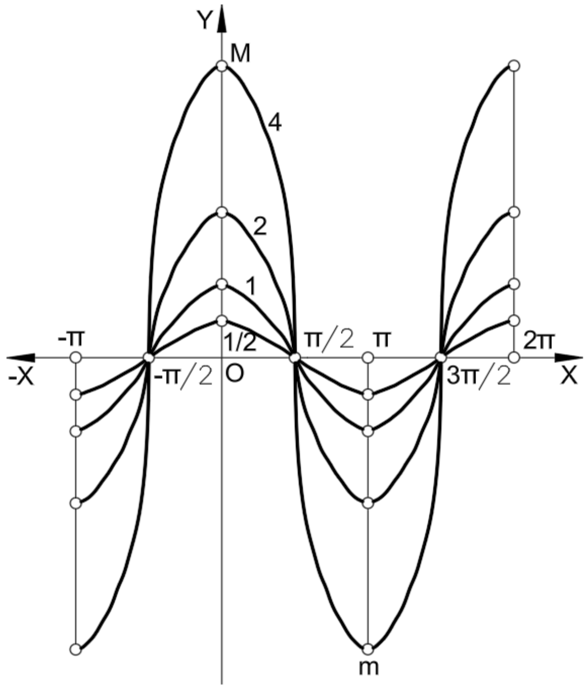

Figure 2 Graph 2 of a Cartesian equation

3. PLOTTING OF CURVES CONVEYED BY MEANS OF AN IMPLICIT CARTESIAN EQUATION

Plane curves can be conveyed by means of an implicit Cartesian equation as follows [4], [7]:

$$f(x,y) = 0 \qquad (4)$$

In order to draw the curve adequate to such an equation few stages must be browsed:

- Analyse of function's symmetry
  If relationship (5) is valid the curve is symmetrical in report to (OX) axis therefore it is enough to draw the part of the curve adequate to positive values of x.

$$f(x,y) = f(-x,y) \qquad (5)$$

  If relationship (6) is valid the curve is symmetrical in report to (OY) axis therefore it is enough to draw the part of the curve adequate to positive values of y.

$$f(x,y) = f(x,-y) \qquad (6)$$

  If relationship (6) is valid the curve is symmetrical in report to the origin of co-ordinates system therefore it is enough to draw the part of the curve adequate to positive values of x and y.

$$f(x,y) = f(-x,-y) \qquad (7)$$

- The first-order derivatives
  The derivative $y'(x)$ is calculated and the minimum and maximum points are found, the points the tangent to the curve is parallel to (OY) axis.
  Proceed analogously for the derivative $x'(y)$ finding the points the tangent to the curve is parallel to (OX) axis.

- The second-order derivative

  The initial equation (4) is homogenized and it is written as follows:

$$f(x,y,z) = 0 \qquad (8)$$

  The partial second-order derivatives in report to x, y, z are calculated, they will provide the co-ordinates of the inflection points (if there are).

- Limits to endpoints

  Function's limits at the endpoints of the increment are calculated. For this purpose, the initial equation (4) is solved replacing x by the values situated and the endpoints of function's definition increment [2], [6].

- Function's variance table and graph

  The results obtained after completing the mentioned stages are gathered in a table in increasing quantities order and there are found the consecutive increments where values of x stay distinctive, continuous, they are not null, so they keep a constant sign. In each of these increments the values and the signs of y are found. Each of these increments are adequate to a curve/arc of curve.

To illustrate the method to draw a function's graph conveyed by means of an implicit Cartesian equation the curves specified by equation (9) (Cassini's ovals) are presented [4]:

$$f(x,y,) = y^4+2\,y^2(x^2+c^2) + (x^2-c^2)^2-a^4 = 0 \qquad (9)$$

In order to draw the curves (Figure 3) the stages previously listed are completed, beginning with the analyse of the symmetry. As relationship (7) is valid the curve is symmetrical in report to both axes *(OX)* and *(OY)* and in report to the origin of the orthogonal system *XOY*. This result allows to study the function only for the positive values of x.

Equation (9) has multiple roots only if [4]:

$$(x^2+c^2)^2 - (x^2-c^2)^2+a^4 = 0 \Rightarrow 4c^2x^2+ a^4 = 0 \qquad (10)$$

or

$$(x^2-c^2)^2- a^4 = 0 \Rightarrow x^2 = c^2 \pm a^2 = 0 \qquad (11)$$

Equation (10) doesn't allow solutions, equation (11) has the following solutions:

$$x_1 = \sqrt{c^2+a^2},\ x_2 = \sqrt{c^2-a^2},\ c \geq a \qquad (12)$$

For solutions (12) $y = 0$, so these points represent the intersection points of the curve with *(OX)* axis.

The first-order derivatives allow finding the maximum (*M*) and minimum (*m*) points, the points the tangents to the curve are parallel to the axes of *XOY* system [8], [9].

$$y' = -\frac{f'x}{f'y} = -\frac{x\,(x^2+y^2-c^2)}{y\,(x^2+y^2+c^2)} \qquad (13)$$

If $y = 0 \Rightarrow y' \rightarrow \pm\infty$

If $x = 0$ and

$$x^2+y^2-c^2 = 0 \Rightarrow y' = 0 \qquad (14)$$

Solving the system consisting of equations (10) and (14) the following solutions are obtained:

$$x^2 = \frac{4c^4-a^4}{4c^2} \Rightarrow x = \frac{\sqrt{4c^4-a^4}}{2c},\ c \geq a \qquad (15)$$
$$y^2 = \frac{a^4}{4c^2} \qquad \Rightarrow y = \frac{a^2}{2c}$$

Table 1

Variance table for the curve specified by equation (9).

x	**y**	**y'**	**curve's points**
0			
$\sqrt{c^2-a^2}$	0	+	A
$\frac{\sqrt{4c^4-a^4}}{2c}$	$\frac{a^2}{2c}$	0	M
$\sqrt{c^2+a^2}$	0	-	B
∞			

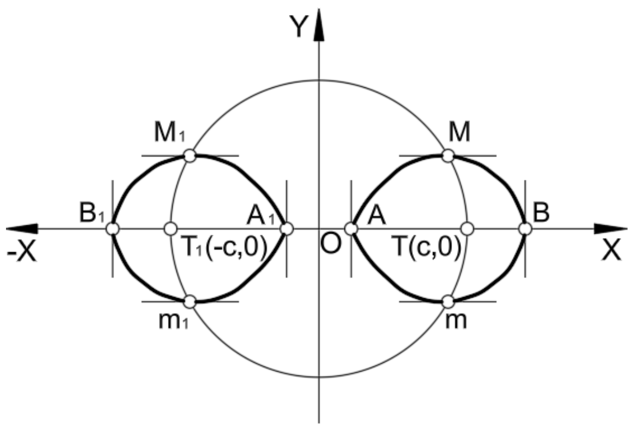

Figure 3 Graph of an implicit Cartesian equation

4. PLOTTING OF CURVES CONVEYED BY MEANS OF A PARAMETRIC EQUATION

Plane curves can be conveyed by means of a parametric equation as follows [4], [7]:

$$\begin{cases} x = x(t) \\ y = y\,(t) \end{cases} \qquad (16)$$

In order to draw the curve adequate to such an equation few stages must be browsed:

- Settle the definition set

The definition set may be an increment or a reunion of increments. Inside this set the limits of parameter *(t)* must be specified.

- Intersection points with the axes
- Analyse the periodicity of the curve

  The curve is periodical if for a positive number T the relationship (17) is valid:

$$\begin{cases} x(t) = x(t+T) \\ y(t) = y\,(t+T) \end{cases} \qquad (17)$$

If the curve is periodic its study can be limited to the study of period *T*, it is not needed the study of the entire definition set.

- Analyse curve's symmetry

  If the equation specifying the curve validates requirement (18), the curve is symmetrical in report to *(OX)* axis; if it validates requirement (19), the curve is symmetrical in report to *(OY)* axis; if it validates requirement (20), the curve is symmetrical in report to the origine; if it validates requirement (21), the curve is symmetrical in report to the first bisector [2], [6].

$$x(t) = x(t') \qquad (18)$$
$$y(t) = - y(t) \qquad (19)$$
$$\begin{cases} x(t') = -x(t) \\ y(t') = y(t) \end{cases} \qquad (20)$$
$$\begin{cases} x(t') = y(t') \\ y(t') = x(t) \end{cases} \qquad (21)$$

- The first-order derivatives

  Calculating these derivatives allow finding the singular points of the curve and of the tangents (if there are). It also must be found the turning points, the points the branches of the curve are located on both sides of the mutual tangent of the two branches.

- Function's variance table and graph

  The results obtained after completing the mentioned stages are gathered in a table in which the maximum points (*M*), the turning points, the tangents are outlined.

To illustrate the method to draw a function's graph conveyed by means of a parametric equation the cycloid is presented.

The cycloid is the curve generated by a point *A* located on a circle rolling along a straight line (Figure 4). To find the parametric equations of the cycloid the following notations are made [3]:

$$\begin{cases} a = CD \\ t = \widehat{ACD} \Rightarrow \widehat{CAE} = 180° - t \end{cases} \qquad (22)$$

$$x = OD - EB = OD - BC = a\,t - a \sin t = a\,(t - \sin t) \qquad (23)$$

$$y = AE = EB + AB = CD + a \cos (180^0 - t) = a - a \cos t = a\,(1 - \cos t) \qquad (24)$$

$$\begin{cases} x = a\,(t - \sin t) \\ y = a\,(1 - \cos t) \end{cases} \qquad (25)$$

The definition set of parameter t is (0, 2π); when the value of t increases with 2π, the value of y doesn't change, but the value of x increases with $2\pi a$. It follows that the cycloid consists of an infinite number of identical arcs of curve, obtained from each other by means of a translation equal to $2k\pi$ along *(OX)* axis [3].

The first-order derivatives:

$$\begin{cases} x' = a\,(1 - \cos t) \\ y' = a \sin t \end{cases} \qquad (26)$$

Table 2

Variance table for the curve specified by equation (25).

t	**x**	**x'**	**y**	**y'**
0	0	0	0	0
π/2	a/2(π–2)	a	a	A
π	πa	2a	2a	0
3π/2	a/2(3π+2)	a	a	-a
2π	2πa	0	0	0

The variance table includes *x, y, x', y'* with $t \in [0, 2\pi]$. In point *M* (πa, $2a$) the curve has a maximum, and points *O* and O_1 represent the turning points, the points the tangents to the curve are *(OY)* axis and the line parallel to this axis located at a distance equal to $2\pi a$ [9].

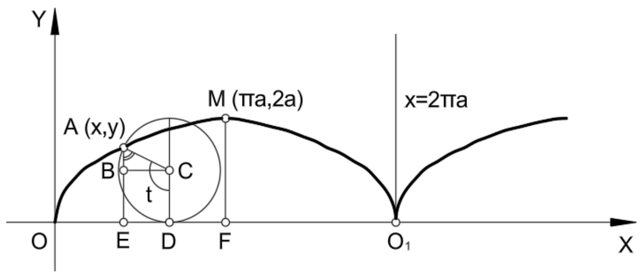

Figure 4 Graph of a parametric equation

5. PLOTTING OF CURVES CONVEYED BY MEANS OF A POLAR EQUATION

Plane curves can be conveyed by means of a Polar equation as follows [4], [7]:

$$r = r\,(\theta) \qquad (27)$$

In order to draw the curve adequate to such an equation few stages must be browsed:

- Settle the definition set

  The definition set represents the increment angle θ must take values in order to get the whole curve. For some polar equations angle θ must take values from real numbers. For equations containing one or more trigonometrical functions, the increment is either a multiple of π or a value added to θ in order not to change the value of r.

- Analyse curve's symmetry

  If the equation specifying the curve validates requirement (28), the curve is symmetrical in report to *(OX)* axis; if it validates requirement (29), the curve is symmetrical in report to *(OY)* axis; if it validates requirement (30), the curve is symmetrical in report to the first bisector [4], [7].

$$r\,(\theta) = r\,(-\theta) \qquad (28)$$
$$r\,(-\theta) = -r\,(\theta) \qquad (29)$$
$$r\,(\theta) = r\,(\pi/2 - \theta) \qquad (30)$$

  The increment set of angle θ may be narrowed considering the mentioned symmetries.

- Intersection points with the axes

  Find the values of θ located inside the incurred increment validating the relationship (31):

$$r\,(\theta) = 0 \qquad (31)$$

- The first-order derivative

Calculating the derivative of the function (r') allows finding the values of angle θ for which $r' = 0$, as well as the signs of the derivative situated near the point the derivative become equal to zero.

- Finding the asymptotes

To find the directions of the asymptotes there must be found the values of the angles for which the function validates the condition [2], [6]:

$$r = r(\alpha) \to \infty \qquad (32)$$

The angles for which equation (32) is true represent the values of the angles between the asymptotes and *(OX)* axis.

- Function's variance table and graph

The results obtained after completing the mentioned stages are gathered in a table allowing plotting the curve.

To illustrate the method to draw a function's graph conveyed by means of a polar equation, the curve specified by equation (33) [6] is presented (Figure 5):

$$r = \frac{a \sin\theta}{2\cos\theta - 1} \qquad (33)$$

As equation (25) only contains trigonometrical functions, the whole curve may be drawn if $\theta \in [0, 2\pi]$.

As relationship (29) is Valid the curve is symmetrical in report to *(OY)* axis. Considering this symmetry, it is enough to assign θ values within increment $[0, \pi]$.

In order to find the asymptotes, it is calculated the value of θ which validate the condition [6]:

$$2\cos\theta - 1 = 0 \qquad (34)$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3} \qquad (35)$$

For the value $\theta = \pi/3$ previously found, an asymptote to function's graph will be plotted.

The first-order derivative:

$$r' = \frac{a \sin^2\theta}{-2 + \cos\theta} \qquad (36)$$

For the value $\theta = \pi/3$ previously found, the derivative becomes equal to $(a/2)$. Therefore, the asymptote adequate to an angle equal to $\pi/3$ is drawn in point P $(a/2, 0)$ (Figure 5).

The variance table includes r and r', with $\theta \in [0, \pi]$.

Table 3

Variance table for the curve specified by equation (25).

θ	**r**	**r'**
0	0	a
π/3	-∞	+∞
π/2	-a	2a
π	2a	a/3

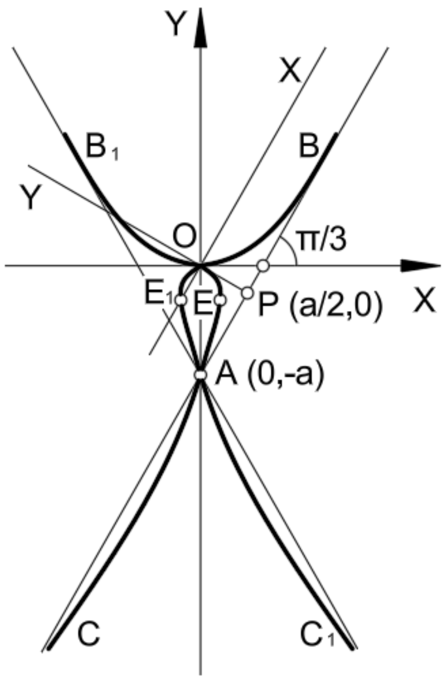

Figure 5 Graph of a polar equation

6 CONCLUSIONS

Plotting curves allow knowing the evolution of phenomena, of technological processes, and at the same time they can estimate their development, the intensity they occur, the maximum and minimum values they can reach. Graphical representations constitute the most flexible method to dialog because of the advantages they offer.

The paper proposes a comparative analysis of the manner plane curves can be plotted based on the equations specifying them from analytical point of view.

The stages that must be completed in order to get to draw the graph are similar no matter what equation specifies the curve:

Settle the definition set of the function which may be an increment or a reunion of increments.

Analyse curve's symmetry in report to orthogonal system's axes, in report to system's origin and in report to the bisectors, as well as the periodicity of the function. This analyse allows to study the function on a certain increment, the rest of the function and of the curve is symmetrically plotted.

Study function's limits at the endpoints at each definition increment, these limits are calculated according the values located at the endpoint of each increment to the argument.

Calculate the first-order derivatives allows estimating curve's profile, if it is increasing, decreasing or constant on different increments. This derivative allows at the same time finding the maximum and minimum points of the curve and of the tangents in these points if they are.

Calculate the second-order derivatives, if necessary, allows estimating the concavity and convexity of the curve as well as the inflection points of the curve, if they are.

Finding the asymptotes, if they are, means to find the values of function's argument for which this function diverges to infinity. The asymptotes may be parallel to co-ordinate system's axes or may make a certain angle in report to the axes.

The values' table gathers the results obtained after completing the stages previously listed and these values are introduced in a table in which the maximum and minimum points, the profile (increasing, decreasing, constant), the inflection points, the asymptotes, the tangents to the curve are revealed.

Drawing the curve's graph is the final stage that translates all the characteristic features of the graph into an orthogonal system *XOY*, connecting all noticeable points of the functions by a continuous curve. The periodicity and the symmetry of the curve must be considered when drawing the graph.

Reviewing the analyse presented by this paper it can be concluded that some manners of analytical convey of a curve are more laborious than others. The same curve can be analytically conveyed by means of several types of equations, therefore before plotting a curve it is recommended to select the equation that allows the easiest method to draw the adequate graph.

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Author(s):

Ph. D., Senior Lecturer Vanda – Ligia DĂNĂILĂ, Technical University "Gheorghe Asachi" din Iaşi, Department of Graphic Communication, E-mail: vanda-ligia.danaila@academic.tuiasi.ro, tel. 0040726332512

Ph. D., Senior Lecturer Irina IGNĂTESCU - MANEA, Technical University "Gheorghe Asachi" din Iaşi, Department of Graphic Communication, E-mail: irina.ignatescu-manea@academic.tuiasi.ro, tel. 004072613129